Shadows and ghosts

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Happy Celebration to you, Manolis, as a tribute to all our interesting discussions and collaborations

Shadows (new) and ghosts (old)
for the gauge-fixing and renormalisation program of SUSY theories

Introduction:

Renormalizing super-Yang–Mills theories is a subtle issue. One must understand the right way of preserving both supersymmetry and gauge invariance, since no regularisation that preserves both supersymmetry and gauge invariance has been found so far.

Basic questions :

- SUSY covariance of Faddeev–Popov ghosts of YM theories.

- Non-existence of a SUSY preserving UV regulator.

- Superfield construction for the perturbation theory only for $N = 1$, with some subtle problems for the gauge-fixing.

- Disentangle gauge and SUSY symmetries.

- SUSY anomalies and SUSY covariant observables.

- Non-renormalisation theorems.

To reach a good set-up at the perturbative level, we introduced new local fields, the shadow fields, which become companions for the Faddeev–Popov ghosts. This came after the discovery of TQFT's, were one must distinguish without ambiguities gauge transformations from general field transformations.

See papers with G. Bossard and Sorella, Berkovits and Bellon.

Gauge-fixing breaks SUSY

Feynman–Landau gauge :

$$\mathcal{L}_{Gauge\ \ fixing} = s(\bar{\Omega}(\partial A + \frac{1}{2}b)) = \frac{1}{2}b^2 + b\partial A - \bar{\Omega}\partial D\Omega \sim -\frac{1}{2}\partial A^2 - \bar{\Omega}\partial D\Omega$$

How $b, \bar{\Omega}, \Omega$ do transform under SUSY ?

The gauge-fixing action breaks the supersymmetry in an unknown way. Even if it occurs through a BRST-exact term, supersymmetry covariance seems lost.

When one fixes the gauge, the complications occurring in SUSY come from

$$\{\delta^{SUSY}_{\epsilon}, \delta^{SUSY}_{\epsilon'}\} = \ \ldots\ldots + \ \delta^{gauge}(\ \epsilon' \ [\ldots\ldots]\epsilon \ )$$

For renormalizing SUSY YM theories, one needs to :

- build Ward-Slavnov-Taylor identities for both SUSY and gauge symmetries,

- classify both SUSY and gauge anomalies,

- find a practical way to adjust counterterms order by ordre in perturbation theory, in a situation where no invariant regulator exists and SUSY is broken by the gauge-fixing term. (An amplification of the situation of e.g. the Weinberg–Salam model in a non-covariant gauge). It is a more difficult question than in the non SUSY cases of QCD or QED, where dimensional regularisation takes care of difficulties. The superfield formalism of N=1 has difficulties because of a lack of power counting.

Generic case

The square of 2 supersymmetry transformations is

$$\delta^{SUSY^2}(\epsilon) \sim \mathcal{L}_\kappa + \delta^{\delta^{gauge}}_{\omega_\epsilon(\varphi)+\kappa^\mu A_\mu}$$

- For an easier notation, ϵ is now a commuting spinor, $\kappa^\mu \equiv \bar{\epsilon}\gamma^\mu\epsilon$, and $\mathcal{L}_\kappa \equiv i_\kappa d + d i_\kappa$ is the Lie derivative. $\omega_\epsilon(\varphi) \sim \bar{\epsilon}\tau^i\varphi_i\epsilon$.

- $\sim$ means modulo equations of motion of a SUSY-invariant action.

- For $N = 1$ and 2, auxiliary fields exist for closing the SUSY algebra, but not for N=4. In this case, by twisting the SUSY, one can isolate sectors with no more than 9 supersymmetries, and work out proofs of finiteness.

- The difficulty one must firstly solve for writing Ward identities and consistency conditions for SUSY is due to the gauge transformations $\delta^{\delta^{gauge}}(\omega_\epsilon(\varphi) + \kappa^\mu A_\mu)$ in the r.h.s. of the closure relations.

Extending the algebra with shadows

$$Q^{SUSY}(\phi) \equiv \delta^{SUSY}_{\epsilon^{0,1}}(\phi) \ + \ \delta^{\delta^{gauge}}_{c^{0,1}}(\phi)$$
$$Q^{SUSY}(c^{0,1}) \equiv \ -\omega_{\epsilon^{0,1}}(\phi) - \kappa^\mu A_\mu - \frac{1}{2}\{c^{0,1}, c^{0,1}\}$$

The square of two transformations Q^{SUSY} is the Lie derivative along $\kappa^\mu = \bar{\epsilon}\gamma^\mu\epsilon$,

$$Q^{SUSY^2}(\epsilon) \sim \mathcal{L}_\kappa$$

The Q^{SUSY} invariance determines the classical SYM Lagrangian,

$$Q^{SUSY} L_{cl} \ = d(...)$$

If we succeed in building s and Q on all relevant fields, including the Faddeev–Popov ghosts, (yet to be introduced), gauge-fixing actions of the type

$$\mathcal{L}_{gf} = sQ^{SUSY}(....)$$

preserves both the SUSY and the BRST symmetry of gauge theory and satisfies locality.

The shadow $c^{0,1}$ has nothing to do with the FP ghost Ω, which has a different grading than $c^{0,1}$ (it is $\Omega^{1,0}$) and cannot mix with it !! We are in fact heading toward a QFT with a ghost and shadow bi-grading.

Compatibility of the algebra of ghosts and shadows

Define the BRST symmetry for the gauge symmetry by the nilpotent operator s. We must have

$$s^2 = 0 \qquad \{s, Q^{SUSY}\} = 0 \qquad {Q^{SUSY}}^2 = \mathcal{L}_\kappa$$

with $\kappa \equiv \bar{\epsilon}\gamma\epsilon$ that is

$$\tilde{d}^2 \equiv (d + s + Q^{SUSY} - i_\kappa)^2 = 0$$

This gives one of these magic formula, analogous to those used e.g. for understanding anomalies, the standard BRST symmetry and the TQFT's, mixed gravitational and gauge symmetries,

$$(d + s^{1,0} + Q^{0,1} - i_\kappa)(A + \Omega^{1,0} + c^{0,1}) + \frac{1}{2}[A + \Omega^{1,0} + c^{0,1}, A + \Omega^{1,0} + \mathfrak{c}^{0,1}] = F + \delta^{SUSY}_{\epsilon^{0,1}}(A) + \omega_{\epsilon^{0,1}}(\phi)$$

Together with its Bianchi identity, it gives all relevant information on the way all fields transform under SUSY and BRST symmetry.

$$
\begin{aligned}
sA &= -D\Omega \\
s\phi &= -[\Omega, \phi] \\
s\Omega &= -\Omega\Omega \\
QA &= \delta_\epsilon^{SUSY}(A) - Dc \\
Q\phi &= \delta_\epsilon^{SUSY}(\phi) - [c, \phi] \\
Qc &= \omega_\epsilon(\phi) + i_\kappa A - cc
\end{aligned} \tag{1}
$$

Moreover, it tells us how (i) SUSY must transform the FP ghosts and (ii) BRST symmetry must transform the shadows. It gives,

$$sc + Q\Omega + [c, \Omega] = 0$$

so we must introduce μ **with**

$$
\begin{aligned}
sc &= \mu \\
s\mu &= 0 \\
Q\Omega &= -\mu - \{\Omega, c\} \\
Q\mu &= -[\omega_\epsilon(\phi), \Omega] - \mathcal{L}_\kappa\Omega - [c, \mu]
\end{aligned}
$$

Antighosts are introduced as BRST quartets, as well as anti-shadows, in order all unphysical fields dont't contribute to the space of observables. Indeed we need a BRST invariant Lagrangian with ghost and shadow number zero. We have

$$\Omega, c, \mu \quad \rightarrow \quad \Omega, c, \mu, \bar{\Omega}, \bar{c}, \bar{\mu}, b$$

with

$$s\bar{\Omega} = b \qquad sb = 0$$
$$s\bar{\mu} = \bar{c} \qquad s\bar{c} = 0$$

$$Q\bar{\Omega} = \mathcal{L}_\kappa \bar{\mu} \qquad Qb = -\mathcal{L}_\kappa \bar{c}$$
$$Q\bar{\mu} = \bar{\Omega} \qquad Q\bar{c} = -b$$

This is a trivial representation space for the algebra

$$s^2 = 0 \qquad \{s, Q^{SUSY}\} = 0 \qquad {Q^{SUSY}}^2 = \mathcal{L}_\kappa$$

It is easy to write a gauge-fixing action that is s-exact and possibly Q-invariant. In general, there will be closed loops of FP-ghost and shadows with various interactions in the generated Feynman diagrams.

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Physics will not depend on the parameters stemming from the s-exact part of the action, provided no anomalies occur.

The gauge-fixed SUSY action is generally

$$\mathcal{L}[A, \phi, \Omega, \bar{\Omega}, b, c, \bar{c}, \mu, \bar{\mu}, \epsilon]$$

It is a local action, and must be completed with source terms for all s and Q transforms of fields,

$$\Sigma = \mathcal{L}_{cal}[A, \phi] + s\Big(\bar{\Psi}^{-1,0}[A, \phi, \Omega, \bar{\Omega}, b, c, \bar{c}, \mu, \bar{\mu}, \epsilon]\Big)$$
$$+\Phi^{s} s\Phi + \Phi^{Q} Q\Phi + \Phi^{sQ} sQ\Phi - \Omega^{s}\Omega\Omega$$
$$+other\ source\ terms\ for\ all\ possible\ s,\ Q\ and\ sQ\ transforms\ of\ fields$$

We introduce α, ξ gauges as :

$$\bar{\Psi}^{-1,0} = \xi\ [\bar{\Omega}(\partial A - \frac{\alpha}{2}b) + \bar{\mu}\partial^2 c] + (1-\xi)\ Q[\bar{\mu}(\partial A - \frac{\alpha}{2}b)]$$

Such ξ, α and ϵ dependent gauges are the most general ones with no quartic ghost interactions. They give a local action that is stable under radiative corrections. We see that the parameters ϵ that appear in the Lagrangian are interpreted as gauge parameters. Physical observables will not depend on them, if no anomaly occur

1) For $\xi = 1$, one has the ordinary FP action with an additive free term

$$L_{GF} = s[\bar{\Omega}(\partial A + \frac{\alpha}{2} b)] + \bar{\mu}\partial^2\mu + \bar{c}\partial^2 c$$

2) For $\xi = 0$, one has a SUSY invariant gauge

$$I_{GF} = \int sQ(\bar{\mu}(\partial A - \frac{\alpha}{2} b)) = s \int [\bar{\Omega}(\partial A - \frac{\alpha}{2} b) + [\bar{\mu}\partial Dc - \partial\delta^{SUSY}(A) + \frac{\alpha}{2}\mu\mathcal{L}_\kappa \bar{c}]]$$

In this SUSY gauge, the shadows propagate and interact.

All these action contains a piece $b\partial A - \frac{\alpha}{2} b^2 - \bar{\Omega}\partial D\Omega$. Those who are familiar with the BRST quantisation will agree that theses gauges are stable and don't generate quartic ghost and shadow interactions, because the antighost and antishadow equations of motions of the tree action as Ward identities can be enforced as Ward identities order by order in perturbation theory.

If no anomaly occurs, one can by definition renormalise and preserve both SUSY and gauge symmetry, as a consequence of the locality. But the possible anomalies can be classified using the descent equations from the magic formula.

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Once both Ward identities for the Green functions of fields and of observables have been established, it is a straightforward (but tedious) task to adjust the counterterms that are necessary to ensure supersymmetry and gauge symmetry at the quantum level. The possibility of that is warranted by having locality and power counting, by having no anomaly, and by the stability of the antighost and antishadow equations of motion. Some of the technical details that one must endure have been detailed within an old paper of Becchi for genuine QCD. The question of not having a regulator that maintains supersymmetry is irrelevant. However, in practice, one wishes to preserve the symmetry of the bare action as much as it is possible, and thus, one uses dimensional reduction regularisation. Minimal subtraction can be tried, but checked and possibly corrected order by order.

Physics

1) Observables $\equiv$ Cohomology of the BRST symmetry s.

2) Thus, the mean values of observable are independent of ϵ, ξ, α.

3) They fall in SUSY multiplets, because there are values of ξ for which SUSY is explicit for the s-exact gauge-fixing term, which Q-exact in this case. The non-SUSY gauges are analogous to those that interpolates gauges between Coulomb and Landau gauge $\partial_0 A_0 + a\partial_i A_i$.

4) Unitarity is a usual : it is hard to prove in detail, but trivial by the heuristic arguments, once there are no anomaly.

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When the gauge parameter ξ is set to 1, the shadow fields $c, \bar{c}, \mu, \bar{\mu}$ become free, since, in this case, their contribution in the action is just $\int Tr(\bar{c}\partial^2 c + \bar{\mu}\partial^2\mu)$. The remaining of the gauge-fixing action is the ordinary Faddeev-Popov action. In this gauge, supersymmetry is not manifest, and supersymmetry covariance of observables is difficult to prove. On the other hand, after having introduced the shadows, one finds that the Landau-Feynman gauges are a continuous limit of more general gauges, which are parametrised by ξ, α and the parameters ϵ of the SUSY transformations. For $\xi = 0$, these gauges are supersymmetric, but one must keep non-zero values for the supersymmetric parameters, and

$$I_{GF} = sQ\int Tr(\bar{\mu}(\partial A - \frac{1}{2}b)$$

The idea is thus to perform the renormalisation programme in the gauge $\xi = 0$, since in this case the supersymmetric Ward identities is simplest. One relies on the independence theorem of the observables upon changes of the gauge parameters, so that their mean values are the same as in the standard FaddeevPopov gauge-fixing, thereby ensuring the supersymmetry covariance, even in this non-SUSY gauge.

SUSY and gauge anomalies

Must be analysed in the SUSY gauge, $\xi = 0$. If no anomalies occur in this gauge, they cannot appear in other gauges.

By definition, anomalies may occur when one renormalise from order $\hbar^{n-1}$ to $\hbar^n$, since local functions $\mathcal{A}$ and $\mathcal{B}$ can show-up in the r.h.s. of Ward identities when one tests the renormalised functional Γ^n,

$$\mathcal{S}^s \Gamma^n = \hbar^n \mathcal{A} \qquad \mathcal{S}^Q \Gamma^n = \hbar^n \mathcal{B}$$

and it may be impossible to absorb $\mathcal{A}$ and $\mathcal{B}$ by the redefinition of local counterterms in the Lagrangian.

The consistency condition is

$$\int s\mathcal{A} = 0 \qquad \int Q\mathcal{B} = 0 \qquad \int (QA + sB) = 0$$

- Equations of motions of ghosts and shadows indicate that $\mathcal{A}$ and $\mathcal{B}$ cannot depend on antighosts and antishadows.

- $\mathcal{A}$ must be the ABBJ anomaly. If it is zero $\mathcal{B}$ must be zero.

- $\mathcal{B}$ possibly non-zero implies that the following relation be valid

$$\int Tr[F \wedge \delta^{SUSY} A \wedge \delta^{SUSY} A + \omega(\phi) F \wedge F] = \delta^{SUSY} (...)$$

One can compute that it can possibly hold only for $N = 1$ SUSY. But this is obvious because chiral fermions only occur for $N = 1$.

Of course, these are known facts. However, by having introduced the shadows, both Ward identities for supersymmetry and gauge invariance allow a safe and convincing verification of the status of gauge and supersymmetry anomalies by the consistency argument, in a recursive construction, at any given finite order of perturbation theory.

Remark : elimination of shadows

In the shadow-Landau gauge (Qs-exact action with $\alpha = 0$), one can functionally eliminate the $c, \bar{c}, \mu, \bar{\mu}$ dependence as the ratio of two equal determinants, given a factor of 1.

But, to express the remaining Q symmetry of the action, one must replace the c dependence of the Q transformations by the result of the elimination of c

$$c \quad \rightarrow c + \frac{1}{D\partial} \partial_\mu \delta^{SUSY}(A)$$

This is a non-local supersymmetry of the Faddeev–Popov gauge-fixed SUSY action in the Landau gauge. It could have been discovered before the introduction of shadows, suggesting a possible reverse construction that eventually makes local this non-local symmetry, leading thereby to shadow fields.

However, a direct introduction of shadow fields may seem more natural. Locality is enforced from the beginning, and one simply solves the question of getting natural SUSY transformation laws of FP ghosts SUSY. Anomalies are then clearly related to ABBJ anomalies. Good and natural SUSY gauges exist.

Conclusion

- FP ghosts and shadows are the proper way for giving local actions for which both SUSY and gauge symmetry are controlled by Ward identities.

- Anomalies and observables are clearly well-defined.

- SUSY covariance is controlled and the issue of using a SUSY invariant UV regulator becomes an irrelevant question, as it it must be. One has a practical method to use dimensional regularisation.

- Convincing proofs of finiteness can be given, from the descent equations for the Q-operator, using $Q^2 = \mathcal{L}_\xi$, and checking its cohomology.

Introduction to twist for SYM, finiteness theorems for N=4 SuperYang–Mills (II)

Shadows provide the SUSY Ward identities. They can be used to demonstrate non-renormalisation theorems.

When the Poincaré supersymmetry closes only on-shell, the proofs about renormalisation are greatly simplified by twisting the spinor fields in tensors, for doing appropriate off-shell closed restrictions of the SUSY algebra.

Moreover, the differential operators s and Q of supersymmetric theories satisfy extended curvature conditions, that are at the origin of SUSY Ward identities for the non-renormalisation theorems.

N=4 action

The gauge invariant action is determined by supersymmetry and $SO(4)\times SU(4)$ global invariance

$$S \equiv \int d^4x \mathbf{Tr}\ (-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi^i D^\mu\phi_i + \frac{i}{2}(\bar\lambda D\lambda) - \frac{1}{2}(\overline{\lambda}[\phi,\lambda]) - \frac{1}{4}[\phi^i,\phi^j][\phi_i,\phi_j])$$

with $\phi \equiv \phi^i\tau_i$ and δ^{Susy} is

$$\delta^{Susy}A_\mu = i(\bar\epsilon\gamma_\mu\lambda) \qquad \delta^{Susy}\phi^i = -(\bar\epsilon\tau^i\lambda) \qquad \delta^{Susy}\lambda = (F + iD\phi + \frac{1}{2}[\phi,\phi])\epsilon$$

To handle the gauge transformations in the closing relations one has the scalar shadow field c valued in the Lie algebra of the gauge group and the differential operator Q made out of δ^{Susy} that is nilpotent modulo a translation and equations of motion and the BRST symmetry operator s.

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The twist

Twist permits one to reduce the size of the SUSY and close Q, so that one and gets rid of equations of motions in the expression of Q^2. We have to go around the fact that fields are arranged in irreducible representations of $Spin(4) \times R_{symmetry}$. Using twisted variables for the spinors in four dimensions, one can in fact reduce the size of the R-symmetry, and get a subalgebra of supersymmetry transformations that close without using equations of motion, and without loosing the information.

The possibility of twisting the $N = 4, d = 4$ theory is seen most easily by going down from from $N = 1, d = 10$, $SO(4) \times U(1) \subset SPIN(7) \subset SO(8) \subset SO(9,1)$.

With 16 on-shell closed supersymmetries, one finds that one cannot get more than 9 generators that close off-shell.

Usually, twist maps spinors on tensors, using covariantly constant spinors for the projection. Its geometrical interpretation changes however, depending on the dimension. In fact, the twist is a baroque operation.

In d=4, the 4 Majorana spinors λ^α and the 6 scalar fields are representation spaces of the R-symmetry $SO(6) \sim SU(4)$. R symmetry. The twist rearranges them as irreducible representations of the following subgroup

$$\begin{aligned} SO(4) \times U(1) &\equiv SU(2)_+ \times \text{diag}[SU(2)_- \times SU(2)_R] \times U(1) \\ &\subset \; SU(2)_+ \times SU(2)_- \times SU(4) \end{aligned}$$

Here

$$SU(2) = \text{diag}[SU(2)_- \times SU(2)_R] \qquad SO(4) = SU(2) \times SU(2)_+$$

So, the $\mathcal{N} = 4$ multiplet has been decomposed in a $SO(4) \times U(1)$ invariant way as 2 independent multiplets with 5 twisted SUSY's

$$A, \lambda, \phi \quad \rightarrow \quad (A_\mu, \Psi_\mu, \eta, \chi^{\mu\nu^-}, \Phi, \bar{\Phi}) \qquad (L, h_{\mu\nu^-}, \bar{\Psi}_\mu, \bar{\eta}, \bar{\chi}_{\mu\nu^-})$$

This is 2 twisted multiplets for $N = 2$. Adding a 6th-SUSY fixes the N=4 multiplet. Here μ is a "twisted world index", which stands for the $(\frac{1}{2}, \frac{1}{2})$ representation of $SO(4) \equiv SU(2)_+ \times SU(2)$.

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The 16 components of the $SU(4)$-Majorana spinors can thus be mapped as $U(1)$ charged $SO(4)$ tensors

$$\lambda \to (\Psi_\mu^{(1)}, \bar{\Psi}_\mu^{(-1)}, \chi_{\mu\nu^-}^{(-1)}, \bar{\chi}_{\mu\nu^-}^{(1)}, \eta^{(-1)}, \bar{\eta}^{(1)})$$

And the 6 scalars ϕ^i in the fundamental representation of $SO(6) \sim SU(4)$ decompose as

$$\phi^i \to (\Phi^{(2)}, \bar{\Phi}^{(-2)}, L^{(0)}, h_{\mu\nu^-}^{(0)})$$

The superscript index states for the $U(1)$ representation.

The 16 SUSY generators and parameters ϵ are respectively twisted into

$$Q^{(1)}, \bar{Q}^{(-1)}, Q_\mu^{(1)}, \bar{Q}_\mu^{(-1)}, Q_{\mu\nu^-}^{(1)}, \bar{Q}_{\mu\nu^-}^{(-1)}$$

$$\epsilon \to (\omega^{(1)}, \varpi^{(-1)}, \varepsilon^{(1)\mu}, \bar{\varepsilon}^{(-1)\mu}, \upsilon^{(1)\mu\nu^-}, \bar{\upsilon}^{(-1)\mu\nu^-})$$

The off-shell closed anti-commutation relations are simplest for the black terms. The definition for the twisted δ^{Susy} is

$$\delta^{Susy} = \varpi Q + \omega\bar{Q} + \bar{\varepsilon}^\mu Q_\mu + \varepsilon^\mu \bar{Q}_\mu + \bar{\upsilon}^{\mu\nu^-} Q_{\mu\nu^-} + \upsilon^{\mu\nu^-} \bar{Q}_{\mu\nu^-}$$

Anticommutation relations are simplest

$$Q^2 = \bar{Q}^2 = \{Q, Q_{\rho\sigma^-}\} = \{Q_\nu, Q_\mu\} = \{Q_{\mu\nu^-}, Q_{\rho\sigma^-}\} = 0 \qquad \{Q, Q_\mu\} = \partial_\mu \qquad \{Q_\mu, Q_{\rho\sigma^-}\} = \epsilon_{\mu\nu\rho\sigma}\partial_\sigma$$

$\bar{Q}$ anticommutes to zero with Q_μ and $Q_{\rho\sigma^-}$.

Remark, the origin of vector supersymmetry

The existence of the vector symmetry, that is, the property that $L = Q(Z)$, with $Q_\mu Z = 0$, is understood from the fact that the energy-momentum tensor being Q-exact,

$$T_{\mu\nu} = QZ_{\mu\nu},$$

the conservation law $\nabla^\nu T_{\mu\nu} = 0$ implies

$$\nabla^\nu Z_{\mu\nu} = 0$$

so that $Z_{\mu\nu}$ must to be the Noether current of a vector symmetry with generator Q_μ.

For $N = 4$, we have

$$\int L_{N=4} = \int QQ_\mu(AdA + \frac{2}{3}A^3 + \kappa\Psi)^\mu = \int \epsilon_{\mu\nu\rho\sigma} Q_\mu Q_\nu Q_\rho Q_\sigma (\Phi^2)$$

Protected operators

Some BPS local operators are protected from renormalisation and their anomalous dimensions vanish to all orders in perturbation theory. In the $N = 4$ theory, these operators have been used to illustrate the AdS/CFT correspondence. Their non-renormalisation properties simplify tests of the conjecture. The $1/2$ BPS primary operators are the gauge-invariant polynomials in the scalar fields of the theory in traceless symmetric representations of the $SO(6)$ R-symmetry group.

It is actually unclear to decide wether their finiteness properties either is a consequence or is at the origin of superconformal invariance.

We will see that the Ward identity for the invariance under a restriction with 5 generators of the Q symmetry is sufficient to prove the important finiteness theorem of $1/2$ BPS operators.

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The off-shell closed four-dimensional supersymmetry with 9 parameters is

$$\delta^{Susy} = \varpi Q + \omega \bar{Q} + \varepsilon^{\mu} \bar{Q}_{\mu} + \upsilon^{\mu\nu^{-}} Q_{\mu\nu^{-}}$$

One has the extended nilpotent differential $d + s + Q - \varpi i_{\varepsilon}$**. The action of** Q **and** s **is**

$$\mathcal{F} \equiv (d + s + Q - \varpi i_{\varepsilon})\big(A + \Omega + c\big) + \big(A + \Omega + c\big)^2 = F + \hat{\mathbf{\Psi}}(\lambda) + \hat{\mathbf{\Phi}}(\phi)$$

with the Bianchi identity that warrantees that $Q^2 = \mathcal{L}_{\varpi i_{\varepsilon}}$

$$(d + s + Q - \varpi i_{\varepsilon})\mathcal{F} + [A + \Omega + c\,,\,\mathcal{F}] = 0$$

with

$$\hat{\mathbf{\Phi}}(\phi) \equiv \varpi^2 \Phi + \omega\varpi L + \varpi \upsilon^{\mu\nu^{-}} h_{\mu\nu^{-}} + (\omega^2 + \varepsilon^{\mu}\varepsilon_{\mu} + \upsilon^{\mu\nu^{-}} \upsilon_{\mu\nu^{-}})\bar{\Phi}$$

For the "gauge choice" $\epsilon^{\mu} = 0$**,** $Q^2 = 0$ **and** $\hat{\mathbf{\Phi}}(\phi)$ **now depends on 5=1+1+3 parameters .**

The Chern–Simon formula says that

$$\mathbf{Tr}\ \mathcal{F}\mathcal{F} = (d + s + Q)\ \mathbf{Tr}\ [\ \mathcal{A}\mathcal{F} - \frac{1}{6}\mathcal{A}^3\ ]$$

By looking at its Lorentz scalar component with shadow number 4, one has identically

$$\begin{aligned}\textbf{Tr}\ \hat{\Phi}(\phi)^2 &\equiv \textbf{Tr}\ [\varpi^2\Phi + \omega\varpi L + \varpi\upsilon^{\mu\nu^-}h_{\mu\nu^-} + (\omega^2 + \upsilon^{\mu\nu^-}\upsilon_{\mu\nu^-})\bar{\Phi}]^2 \\ &= Q\ \textbf{Tr}\ [\ c(\varpi^2\Phi + \omega\varpi L + \varpi\upsilon^{\mu\nu^-}h_{\mu\nu^-} + (\omega^2 + \upsilon^{\mu\nu^-}\upsilon_{\mu\nu^-})\bar{\Phi}) - \frac{1}{6}c^3\ ]\end{aligned}$$

The Feynman rules imply that all possible insertions in the partition function of the Q-antecedent of Tr $\hat{\Phi}(\phi)^2$, that is all local quadratic operators contained in Tr $[c(\varpi^2\Phi+\omega\varpi L+\varpi\upsilon^{\mu\nu^-}h_{\mu\nu^-}+(\omega^2+\upsilon^{\mu\nu^-}\upsilon_{\mu\nu^-})\bar{\Phi})-\frac{1}{6}c^3)$, are finite to all orders in perturbation theory. Since the Q symmetry is preserved by the theory, the insertions of the term in the left hand-side of the equation are finite too, which proves the assertion. The proof extends to any invariant polynomial of $\mathcal{P}\hat{\Phi}(\phi)$.

By expanding Tr $\hat{\Phi}(\phi)^2$, one gets indeed the 20 operators as coefficient of the quadratic form in the 5 =1+1+3 parameters of the scalar and vector twisted supersymmetry.

$$\begin{aligned}&\textbf{Tr}\ \big(\Phi^2\big)\,,\ \textbf{Tr}\ \big(\Phi L\big)\,,\ \textbf{Tr}\ \big(\Phi\bar{\Phi} + \frac{1}{2}L^2\big)\,,\ \textbf{Tr}\ \big(\bar{\Phi}L\big)\,,\ \textbf{Tr}\ \big(\bar{\Phi}^2\big)\,,\\ &\textbf{Tr}\ \big(\Phi h_I\big)\,,\ \textbf{Tr}\ \big(L h_I\big)\,,\ \textbf{Tr}\ \big(\bar{\Phi}h_I\big)\,,\ \textbf{Tr}\ \big(\delta_{IJ}\Phi\bar{\Phi} + \frac{1}{2}h_I h_J\big)\end{aligned}$$

So, finiteness is nothing but a simplest consequence of the Chern–Simon formula to invariant polynomials of $\hat{\Phi}(\phi)$, using $Q^2 = 0$. The proof is completely analogous to that of the ABBJ theorem.

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Contact with the usual statement is done by showing that any given invariant polynomial $\mathcal{P}(\hat{\mathbf{\Phi}})$ for $\epsilon^\mu = 0$ gives gauge-invariant polynomials in the scalar fields of the theory in traceless symmetric representations of the $SO(6)$ R-symmetry group.

Cancellation of the β function from descent equations

Showing that the coupling constant of the $N = 4$ theory is not rescaled by renormalisation, amounts to show that the action $S = \int \mathcal{L}_4^0$ has vanishing anomalous dimension, in the sense that it cannot be renormalized by anything but a mixing with a BRST-exact counterterms. To prove it, we can use the fact that descent equations imply that the Lagrangian density is uniquely linked to a combination of protected operators, with coefficients that are fixed functions of the supersymmetric parameters.

As said before, the reduced supersymmetry with the six generator Q, $\bar{Q}$ and $\bar{Q}_\mu$ is sufficient to completely determine the classical action. For simplicity, we will thus restrict δ^{Susy} to these generators, taking the gauge $\upsilon^{\mu\nu^-} = 0$. Because $\mathcal{L}_4^0$ (and $Ch_4^0 = Tr(FF)$) are supersymmetric invariant only modulo a boundary-term, the algebraic Poincaré lemma predicts series of cocycles, which are linked to $\mathcal{L}_4^0$ and Ch_4^0 by descent equations, as follows:

$$\begin{aligned}
\delta^{Susy}\mathcal{L}_4^0 + d\mathcal{L}_3^1 &= 0 \\
\delta^{Susy}\mathcal{L}_3^1 + d\mathcal{L}_2^2 &= \varpi i_\varepsilon \mathcal{L}_4^0 \\
\delta^{Susy}\mathcal{L}_2^2 + d\mathcal{L}_1^3 &= \varpi i_\varepsilon \mathcal{L}_3^1 \\
\delta^{Susy}\mathcal{L}_1^3 + d\mathcal{L}_0^4 &= \varpi i_\varepsilon \mathcal{L}_2^2 \\
\delta^{Susy}\mathcal{L}_0^4 &= \varpi i_\varepsilon \mathcal{L}_1^3
\end{aligned}$$

One can compute

$$\mathcal{L}_0^4 = \frac{1}{2}\mathbf{Tr}\left(\left(\varpi^2\Phi + \varpi\omega L + \omega^2\bar{\Phi}\right)^2 + \varpi^2|\varepsilon|^2\bar{\Phi}^2\right)$$

All these operator are finite, simply because the last cocycle $\mathcal{L}_0^4$ is a linear combination of the protected operators found before, and thus, its anomalous dimension is zero. This permits to prove that its ascendant $\mathcal{L}_4^0$ can only be renormalized by d-exact or $\mathcal{S}_{|\Sigma}$-exact counterterms.

Clearly, for the sake of understanding $N = 4$ SUSY, the extended form

$$\tilde{\mathcal{L}}_4 \equiv \mathcal{L}_4^0 + \mathcal{L}_3^1 + \mathcal{L}_2^2 + \mathcal{L}_1^3 + \mathcal{L}_0^4$$

as well as its partner $\tilde{C}h_4^0$ are interesting objects.

....

$$Ch = \frac{1}{2}\mathbf{Tr}\ \left(F + \varpi\Psi + \omega\bar{\Psi} + g(\varepsilon)\eta + g(J_I\varepsilon)\chi^I + \varpi^2\Phi + \varpi\omega L + (\omega^2 + |\varepsilon|^2)\bar{\Phi}\right)^2 \tag{2}$$

is $(d + \delta^{Susy} - \varpi i_\varepsilon)$ **invariant.**

Conclusion

The set of fields of a supersymmetric theory has been extended from simple principles.

With the introduction of shadow fields, one can express supersymmetry under the form of a nilpotent differential operator.

By twisting the spinor and scalar fields, one find subalgebra of supersymmetry with no equations of motions in the closure relations.

This simplifies all proofs of various renormalisation theorems, e.g. for the $N = 4$ SYM theory.

On can question the relevance of using too many supersymmetries, some of them being for free.

In supergravity, similar features occur. By using the reparametrization invariance and the invariance under a smaller subalgebra of rigid supersymmetry allowed by twist, analogous phenomena show up.

Anyway, happy Celebration, Manolis, I wish you the best.