Moment Analysis of Hadronic Vacuum Polarization
Proposal for a lattice QCD evaluation of $g_\mu - 2$

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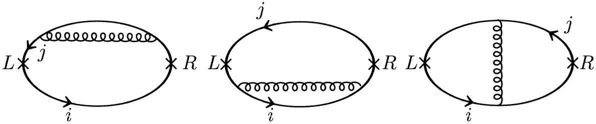
FloratosFest2014
New Horizons in Particles, Strings and Membranes

Left-Right Correlation Function (VV-AA) in QCD



$\Delta^{(1)}(q^2)_{i,j}$	Transverse Component
$\Delta^{(0)}(q^2)_{i,j}$	Longitudinal Component

i and j are flavour indices and m_i and m_j quark masses.

First Weinberg Sum Rule Converges even with explicit breaking

$$\left[\Delta^{(1)}(q^2)_{i,j} + \Delta^{(0)}(q^2)_{i,j}\right]_{-q^2\to\infty} = \frac{\alpha_s}{\pi}\frac{1}{\pi^2}\frac{m_i m_j}{q^2} + \frac{\alpha_s}{\pi}\,\mathcal{O}\left[\left(\frac{m^2}{q^2}\right)^2 \log\frac{m^2}{q^2}\right]$$

Motivation of today's Talk

$a_\mu(\mathrm{E821-BNL}) = 116\ 592\ 089(54)_{\mathrm{stat}}(33)_{\mathrm{syst}} \times 10^{-11}[0.54\mathrm{ppm}]$

Future Experiments:
FNAL with $\pm$0.14 ppm overall uncertainty
JPARC with similar uncertainty but very different technique

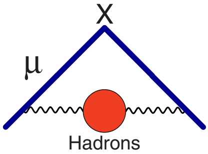
Standard Model Contributions

J.P. Miller, E. de Rafael, B.L. Roberts, D. Stöckinger, Annu. Rev. Part. Nucl. Phys. '12

CONTRIBUTION	RESULT IN 10^{-11} UNITS
QED (leptons)	$116\ 584\ 718.85 \pm 0.04$
HVP(lo)[e^+e^-]	$6\ 923 \pm 42$
HVP(ho)	-98.4 ± 0.7
HLxL	105 ± 26
EW	153 ± 1
Total SM	$116\ 591\ 801 \pm 49$

Persistent 3.6σ discrepancy between SM theory and Experiment

HVP Contribution to the Muon Anomaly



Muon Anomaly from HVP

Standard Formulation in terms of the Hadronic Spectral Function

$$\frac{1}{2}(g_\mu - 2)_{\mathrm{Hadrons}} \equiv a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt}{t}\int_0^1 dx\,\frac{x^2(1-x)}{x^2+\frac{t}{m_\mu^2}(1-x)}\frac{1}{\pi}\mathrm{Im}\Pi(t)$$

where

$$\sigma(t)_{[e^+e^-\to(\gamma)\to\mathrm{Hadrons}]} = \frac{4\pi^2\alpha}{t}\frac{1}{\pi}\mathrm{Im}\Pi(t)$$

Euclidean Formulation (*Lautrup- de Rafael '69*)

In lattice QCD it is necessary to go Euclidean

$$\Pi(Q^2) = \int_0^\infty \frac{dt}{t} \frac{-Q^2}{t+Q^2} \frac{1}{\pi} \mathrm{Im}\Pi(t)\,, \qquad \text{with euclidean} \quad Q^2 = \frac{x^2}{1-x} m_\mu^2\,.$$

HVP in Lattice QCD

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi} \int_0^1 dx\,(1-x) \int_0^\infty \frac{dt}{t} \frac{\frac{x^2}{1-x} m_\mu^2}{t + \frac{x^2}{1-x} m_\mu^2} \frac{1}{\pi} \mathrm{Im}\Pi(t)\,,$$

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \left[-\Pi \left(\frac{x^2}{1-x} m_\mu^2 \right) \right].$$

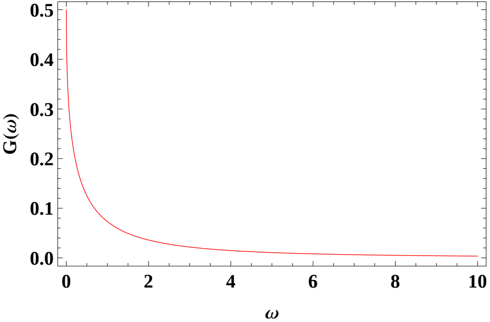
$$\text{Set} \quad \omega = \frac{Q^2}{m_\mu^2} = \frac{x^2}{1-x}\,,$$

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi} \int_0^\infty \frac{d\omega}{\omega} \frac{1}{4} \left[(2+\omega) \left(2 + \omega - \sqrt{\omega}\sqrt{4+\omega} \right) - 2 \right] \left(-\omega \frac{d}{d\omega} \Pi \left(\omega m_\mu^2 \right) \right)$$

Comment on Lattice QCD Evaluations

$$a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_0^\infty \frac{d\omega}{\omega}\,\frac{1}{4}\left[(2+\omega)\left(2+\omega-\sqrt{\omega}\sqrt{4+\omega}\right)-2\right]\left(-\omega\frac{d}{d\omega}\Pi\left(\omega m_\mu^2\right)\right)$$

$$G(\omega) = \frac{1}{4}\left[(2+\omega)(2+\omega-\sqrt{\omega}\sqrt{4+\omega})-2\right]$$



So far, Lattice QCD evaluations need extrapolations with
Models or Padé Approximants

Moment Analysis (Model Independent)

Mellin-Barnes Integral Representation of a_μ^{HVP}

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \mathcal{F}(s)\, \underbrace{\mathcal{M}(s)}, \qquad \mathrm{Re}\ c \in \,]0, +1[$$

$$\mathcal{F}(s) = -\Gamma(3-2s)\Gamma(-3+s)\Gamma(1+s)$$

$$\mathcal{M}(s) = \underbrace{\int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi} \mathrm{Im}\Pi(t)}_{\textit{Mellin Transform of the Spectral Function}}$$

Useful representation to extract the asymptotic expansion for $\frac{m_\mu^2}{t} < 1$.

Expansion in Moment Approximants

$$
\begin{aligned}
a_\mu^{\mathrm{HVP}} &= \left(\frac{\alpha}{\pi}\right)\left\{\frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1)\right. \\
&+ \frac{97}{10}\mathcal{M}(-2) + 6\tilde{\mathcal{M}}(-2) \\
&+ \left.\frac{208}{5}\mathcal{M}(-3) + 28\tilde{\mathcal{M}}(-3) + \mathcal{O}\left[\tilde{\mathcal{M}}(-4)\right]\right\}
\end{aligned}
$$

Two types of Moments

Normal Moments:

$$
\mathcal{M}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1+n}\frac{1}{\pi}\mathrm{Im}\Pi(t)\,, \quad n = 0, 1, 2, \ldots
$$

Log weighted Moments:

$$
\tilde{\mathcal{M}}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1+n}\log\frac{m_\mu^2}{t}\,\frac{1}{\pi}\mathrm{Im}\Pi(t)\,, \quad n = 1, 2, 3, \cdots
$$

The Moment Approximants in a Phenomenological Toy Model

$$a_\mu^{\rm HVP}(e^+e^-) = (6.923 \pm 0.042) \times 10^{-8} \quad (0.6\%)$$

M. Davier et al' 10

$$a_\mu^{\rm HVP}(\text{toy model}) = 6.936 \times 10^{-8}$$

D. Bernecker and H.B. Meyer, '11; L. Lelllouch, '14

Numerical Values of the Moment Approximants (Toy Model)

$$\left(\frac{\alpha}{\pi}\right)\frac{1}{3}\mathcal{M}(0) = 8.071 \times 10^{-8} \quad (16\%)$$

$$\left(\frac{\alpha}{\pi}\right)\left[\frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1)\right] = 7.240 \times 10^{-8} \quad (4\%)$$

$$\left(\frac{\alpha}{\pi}\right)\left[\frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1) + \frac{97}{10}\mathcal{M}(-2) + 6\tilde{\mathcal{M}}(-2)\right] = 7.022 \times 10^{-8} \quad (1\%)$$

Fourth Approximation is already within 0.4% of toy model result

The Moment Approximants in Lattice QCD

The Leading Moment is a rigorous upper bound

J.S. Bell-de Rafael '69

$$a_{\mu}^{\rm HVP} < \left(\frac{\alpha}{\pi}\right)\frac{1}{3}\underbrace{\int_{4m_{\pi}^2}^{\infty}\frac{dt}{t}\,\frac{m_{\mu}^2}{t}\,\frac{1}{\pi}\,{\rm Im}\Pi(t)}_{\mathcal{M}(0)} = \left(\frac{\alpha}{\pi}\right)\frac{1}{3}\underbrace{\left(-m_{\mu}^2\frac{d}{dQ^2}\Pi(Q^2)\right)_{Q^2=0}}_{\text{Lattice QCD}}$$

- Overestimates $a_{\mu}^{\rm HVP}$ by less than 20% (not bad for a rigorous bound)
- The slope of $\Pi\left(Q^2\right)$ at the origin (r.h.s.) can be evaluated in lattice QCD
- It is difficult to imagine that, unless lattice QCD does better than phenomenology in this simple case, it will ever reach a competitive accuracy of the full determination of $a_{\mu}^{\rm HVP}$.

$\mathcal{M}(-n)$ Moments correspond to successive derivatives of $\Pi(Q^2)$ at $Q^2 = 0$

$$\underbrace{\mathcal{M}(-n)}_{n=0,1,2\dots} = \int_{4m_{\pi}^2}^{\infty}\frac{dt}{t}\left(\frac{m_{\mu}^2}{t}\right)^{1+n}\frac{1}{\pi}{\rm Im}\Pi(t) = \frac{(-1)^{n+1}}{(n+1)!}(m_{\mu}^2)^{n+1}\left(\frac{\partial^{n+1}}{(\partial Q^2)^{n+1}}\Pi(Q^2)\right)_{Q^2=0}$$

These derivatives can be determined in Lattice QCD

The Log Weighted Moments in Lattice QCD

$$\tilde{\mathcal{M}}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^n \log \frac{m_\mu^2}{t} \frac{1}{\pi} \mathrm{Im}\Pi(t)$$

They require the evaluation of integrals of the type

Integrals in the Euclidean to be evaluated in lattice QCD

$$\Sigma(-n) \equiv \int_{4m_\pi^2}^{\infty} dQ^2 \left(\frac{m_\mu^2}{Q^2}\right)^{n+1} \left(-\frac{\Pi(Q^2)}{Q^2}\right) \quad n = 1, 2, 3 \dots$$

Example:

$$\tilde{\mathcal{M}}(-1) = -\log \frac{4m_\pi^2}{m_\mu^2} \underbrace{\mathcal{M}(-1)}_{\text{Latt. QCD}} + \underbrace{\Sigma(-1)}_{\text{Latt. QCD}} - \frac{m_\mu^2}{4m_\pi^2} \underbrace{\mathcal{M}(0)}_{\text{Latt. QCD}} + \mathcal{O}\left[\mathcal{M}(-2)\right]$$

- Contrary to the evaluation of a_μ^{HVP}, the Euclidean moments $\Sigma(-1)$, $\Sigma(-2)$, ... are not weighted by a heavily peaked kernel at small Q^2.
- The threshold of integration is at a rather large value $Q^2 = 4m_\pi^2$ instead of zero.
- The determination of these Euclidean moments in lattice QCD and their comparison with the corresponding phenomenological expressions in terms of the hadronic spectral function, provide valuable further tests.

Successive Aproximations to $a_\mu^{\rm HVP}(\text{phen. model}) = 6.936 \times 10^{-8}$

Quantities to be evaluated in lattice QCD

$$\underbrace{\mathcal{M}(0)}_{10.424}\,;\quad \underbrace{\Sigma(-1)}_{1.223}\,,\quad \underbrace{\mathcal{M}(-1)}_{0.278}\,;\quad \underbrace{\Sigma(-2)}_{0.113}\,,\quad \underbrace{\mathcal{M}(-2)}_{0.012}\quad \text{and}\quad \underbrace{\mathcal{M}(-3)}_{0.001}$$

Final Results (numbers are those from the Phenomenological Toy Model)

- **1st Approximation** (same as Bell-de Rafael):

$$\left(\frac{\alpha}{\pi}\right)\frac{1}{3}\mathcal{M}(0) = 8.071 \times 10^{-8}$$

- **2nd Approximation**

$$\left(\frac{\alpha}{\pi}\right)\left\{\left(\frac{1}{3}-\frac{m_\mu^2}{4m_\pi^2}\right)\mathcal{M}(0)+\left(\frac{25}{12}-\log\frac{4m_\pi^2}{m_\mu^2}\right)\mathcal{M}(-1)+\Sigma(-1)+\frac{4m_\pi^2}{m_\mu^2}\mathcal{M}(-2)\right\}$$
$$= 7.265(34) \times 10^{-8}$$

- **3rd Approximation**

$$\left(\frac{\alpha}{\pi}\right)\left\{\left(\frac{1}{3}-\frac{m_\mu^2}{4m_\pi^2}+3\left(\frac{m_\mu^2}{4m_\pi^2}\right)^2\right)\mathcal{M}(0)+\left(\frac{25}{12}-\log\frac{4m_\pi^2}{m_\mu^2}-6\frac{m_\mu^2}{4m_\pi^2}\right)\mathcal{M}(-1)\right.$$
$$\left.+\left(\frac{97}{10}-6\log\frac{4m_\pi^2}{m_\mu^2}+\frac{4m_\pi^2}{m_\mu^2}\right)\mathcal{M}(-2)+\Sigma(-1)-6\Sigma(-2)+\frac{4m_\pi^2}{m_\mu^2}\left(6-\frac{1}{2}\frac{4m_\pi^2}{m_\mu^2}\right)\mathcal{M}(-3)\right\}$$
$$= 7.027(6) \times 10^{-8}\,.$$

The *moment analysis* approach may gradually lead to an accurate determination of a_μ^{HVP}, providing at the same time many tests of *lattice QCD evaluations* to be confronted with phenomenological determinations using experimental data.

Dear Manolis

Welcome to the club of Active Retired Theorists!

The *moment analysis* approach may gradually lead to an accurate determination of a_{μ}^{HVP}, providing at the same time many tests of *lattice QCD evaluations* to be confronted with phenomenological determinations using experimental data.

Dear Manolis

Welcome to the club of Active Retired Theorists!

Expansion in terms of Moments

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \mathcal{F}(s)\, \underbrace{\mathcal{M}(s)}, \qquad \mathrm{Re}\ c \in\ ]0, +1[$$

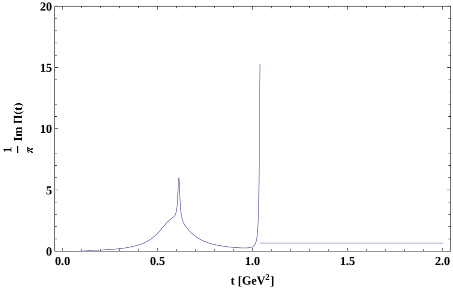
$$\mathcal{F}(s) = -\Gamma(3-2s)\Gamma(-3+s)\Gamma(1+s) \quad \text{and} \quad \underbrace{\mathcal{M}(s) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi} \mathrm{Im}\Pi(t)}_{Mellin\ Transform\ of\ Spectral\ Function}$$

$\mathcal{M}(s)$ in QCD is finite for $s < 1$ and singular at $s = 1$ (*with known residue from pQCD*)

Singular expansion of $\mathcal{F}(s)$:

$$\mathcal{F}(s) \asymp \frac{1}{3}\frac{1}{s} - \frac{1}{(s+1)^2} + \frac{25}{12}\frac{1}{s+1} - \frac{6}{(s+2)^2} + \frac{97}{10}\frac{1}{s+2} - \frac{28}{(s+3)^2} + \frac{208}{5}\frac{1}{s+3} + \cdots$$

Hadronic Spectral Function



D. Bernecker and H.B. Meyer, '11; L. Lellouch, '14

- The most recent experimental determination from e^+e^- data:

$$a_\mu^{\mathrm{HVP}} = (6.923 \pm 0.042) \times 10^{-8}\,.$$

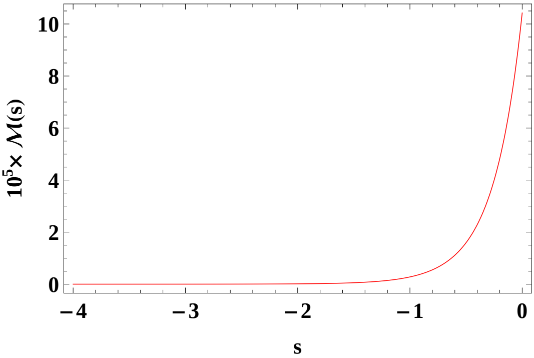
- This phenomenological parametrization:

$$a_\mu^{\mathrm{HVP}} = 6.936 \times 10^{-8}\,.$$

Mellin Transform in the Phenomenological Toy Model

$$\underbrace{\mathcal{M}(s) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi}\mathrm{Im}\Pi(t)}_{\text{Mellin Transform of Spectral Function}}$$

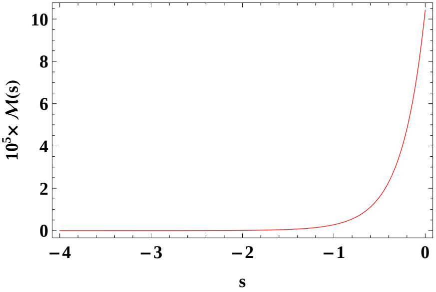
$$\mathcal{M}_{\rm pQCD}(s)|_{s->1} \Rightarrow \left(\frac{\alpha}{\pi}\right)\left(\frac{2}{3}\right) N_c \, \frac{1}{3} \, \frac{1}{1-s} \, .$$

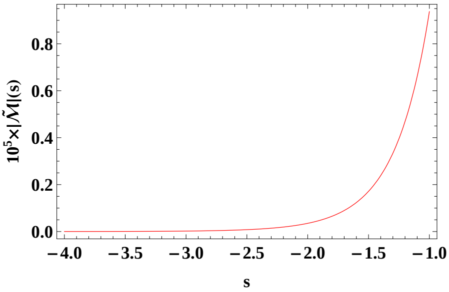


The Mellin Transform of $\frac{1}{\pi}\mathrm{Im}\Pi(t)$ is a Monotonously Decreasing Function

Mellin Transform and Derivative of Mellin Transform in Toy Model

$$\mathcal{M}(s) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi}\mathrm{Im}\Pi(t)\,, \quad \tilde{\mathcal{M}}(s) = -\frac{d}{ds}\mathcal{M}(s) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \log\frac{m_\mu^2}{t}\, \frac{1}{\pi}\mathrm{Im}\Pi(t)$$





Both Mellin Transforms of $\frac{1}{\pi}\mathrm{Im}\Pi(t)$ are decreasing smooth Functions

The Log Weighted Moments in Lattice QCD

First do a change of scale:

$$\tilde{\mathcal{M}}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^n \log\frac{m_\mu^2}{t}\,\frac{1}{\pi}\mathrm{Im}\Pi(t)$$

$$\tilde{\mathcal{M}}(-n) = -\log\frac{4m_\pi^2}{m_\mu^2}\mathcal{M}(-n) + \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^n \log\frac{4m_\pi^2}{t}\,\frac{1}{\pi}\mathrm{Im}\Pi(t)$$

Construct the Euclidean Integral $\Sigma(-1)$

$$\Sigma(-1) \equiv \underbrace{\int_{4m_\pi^2}^{\infty} dQ^2\left(\frac{m_\mu^2}{Q^2}\right)^2\left(-\frac{\Pi(Q^2)}{Q^2}\right)}_{\text{Lattice QCD}} = \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^2 \log\frac{4m_\pi^2}{t}\,\frac{1}{\pi}\mathrm{Im}\Pi(t)$$

$$+\frac{m_\mu^2}{4m_\pi^2}\mathcal{M}(0) - \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^2 \log\left(1+\frac{4m_\pi^2}{t}\right)\,\frac{1}{\pi}\mathrm{Im}\Pi(t)$$

Notice the relation:

$$\tilde{\mathcal{M}}(-1) = -\log\frac{4m_\pi^2}{m_\mu^2}\mathcal{M}(-1) + \underbrace{\Sigma(-1)}_{\text{Latt. QCD}} - \frac{m_\mu^2}{4m_\pi^2}\mathcal{M}(0) + \frac{4m_\pi^2}{m_\mu^2}\mathcal{M}(-2) + \cdots$$

The Log Weighted Moments in Lattice QCD (continuation)

Construct Next the Euclidean Integral $\Sigma(-2)$ (with one more $\frac{m_\mu^2}{Q^2}$ power)

$$\Sigma(-2) \equiv \underbrace{\int_{4m_\pi^2}^{\infty} dQ^2 \left(\frac{m_\mu^2}{Q^2}\right)^3 \left(-\frac{\Pi(Q^2)}{Q^2}\right)}_{\text{Lattice QCD}} = -\int_{4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^3 \log\frac{4m_\pi^2}{t}\,\frac{1}{\pi}\mathrm{Im}\Pi(t)$$

$$+\frac{1}{2}\left(\frac{m_\mu^2}{4m_\pi^2}\right)^2 \mathcal{M}(0) - \frac{m_\mu^2}{4m_\pi^2}\mathcal{M}(-1) + \int_{4m_\pi^2}^{\infty}\frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^3 \log\left(1+\frac{4m_\pi^2}{t}\right)\frac{1}{\pi}\mathrm{Im}\Pi(t)$$

The relation to the wanted $\tilde{\mathcal{M}}(-2)$ is then:

$$\tilde{\mathcal{M}}(-2) = -\log\frac{4m_\pi^2}{m_\mu^2}\mathcal{M}(-2) - \underbrace{\Sigma(-2)}_{\text{Latt. QCD}} + \frac{1}{2}\left(\frac{m_\mu^2}{4m_\pi^2}\right)^2\mathcal{M}(0) - \frac{m_\mu^2}{4m_\pi^2}\mathcal{M}(-1) + \frac{4m_\pi^2}{m_\mu^2}\mathcal{M}(-3) + \cdots$$

COMMENTS ON THE HLbyLS CONTRIBUTION

HLbyLS Contribution to the Muon Anomaly in Chiral Limit with $M \to \infty$

$$a_\mu^{(\mathrm{HLbyLS})} = \underbrace{\left(\frac{\alpha}{\pi}\right)^3 N_c^{\,2}\frac{m_\mu^2}{16\pi^2 f_\pi^2}\left[\frac{1}{3}\ln^2\frac{M}{m_\pi} + \mathcal{O}\left(\ln\frac{M}{m_\pi}\right) + \mathcal{O}(1)\right]}_{\text{Knecht}-\text{Nyffeler}-\text{Perrottet}-\text{de Rafael}'02} + \left(\frac{\alpha}{\pi}\right)^3 N_c\,\mathcal{O}\left(\frac{m_\mu^2}{M^2}\right)$$

- Clearly, in this limit, the $\ln^2\frac{M}{m_\pi}$ term dominates.
- Once m_μ^2 factored out, -as it should- the pion mass becomes an infrared cut-off.
- However, m_π finite (and not much larger than m_μ).
  Also $M \simeq M_\rho$ (not large enough).
- Therefore, in practice one has to worry about $\mathcal{O}\left(\frac{m_\mu^2}{m_\pi^2}\right)$ corrections and, because of the *physical hadronic spectrum and couplings*, the $\ln^2\frac{M_\rho}{m_\pi}$ dominance becomes questionable.
- Also, subleading corrections in $1/N_c$ (*pion-loop contribution*) may become relevant.